CS 6002
Selected Areas of mechanism design



Quiz 1, 2 - 15%.
midsem - 30%.
project-
presentation-15%.
Report - 25%.

game theory by
Michael Machler
Yoav Shoham - Multiagent
systems

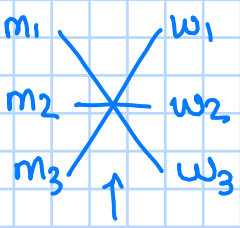
5th Jan:

Two sided market:

preferences on both sides, we have to construct a matching scheme

2 group of agents, a matching is stable if there is no pair of agents (one from each side) who would prefer to be matched with each other instead current matches

$w_1 > w_2 > w_3$
$w_2 > w_1 > w_3$
$w_1 > w_3 > w_2$



stable matching

$m_3 > m_2 > m_1$
$m_2 > m_3 > m_1$
$m_3 > m_2 > m_1$

Fair division:

dividing resource among individuals in a way that is perceived as fair by all participants

Heterogenous goods may hold different level of value

normalisation: $v_i([0,1]) = 1$ ← value function

$\forall$ agent: $v_i(A_i) \geq \frac{1}{n}$, $v_i([0,1]) = 1$

Equality or each agent gets atleast the average share

Envy-free: $\forall i, j \quad i \neq j, \; v_i(A_i) \geq v_i(A_j)$ ← j's share
↑ from i's prespective

and $\forall_i \sum_{j \in N} v_i(A_i) \geq \sum_{j \in N} v_i(A_j) = v_i(\bigcup_{j \in N} A_j) = 1$

$\Rightarrow n v_i(A_i) \geq 1$

$\Rightarrow v_i(A_i) \geq \frac{1}{n}$

Note: Envy-free (EF) property $\Rightarrow$ Proportionality (PROP)

Eg: allocation of a cake, for N=2 we get the one cut, other choose strategy

We will mainly look at:

1. Stable matching
2. Fair division/Allocation
3. Cooperative games

One sided matching:

assigning items (eg goods/task/slot) to agents where only agent have pref over items, items do not have preferences

Eg: House allocation, where money is not allowed and only option is to exchange/relocate

$M = \{a_1, \ldots a_m\}$ set of objects

$N = \{1, 2 \ldots n\}$ $m \geq n$ set of agents

Each agent has a linear order over the object

types of linear order:

① complete: $\forall a, b \in M$, either aRb or bRa

② Transitive: aRb and $bRc \Rightarrow aRc$

③ antisymmetric: aRb and $bRa \Rightarrow a = b$

P denotes preference order of agent i

$P = \{P_1 \ldots P_n\}$ preference profile

μ = set of possible linear orders over M, $|\mu| = m!$

$P_i(k, S)$: k^{th} top object in $S \subseteq M$, in P_i

Note: Aim is to find collective decision problem satisfying SP, PE, etc

Defn: (SP) strategy proofness, An SCF $f: U^n \to A$ is SP if

$f_i(P_i, P_{-i})\ P_i\ f(P_i', P_{-i})$ or $f_i(P_i, P_{-i}) = f_i(P_i', P_{-i})$

$\forall P_i, P_{-i}, P_i', \forall i \in N$

Defn: (PE) An SCF f is pareto efficient if $\forall$ possible P, $\nexists$ any other matching $a \in A$, $a \neq f(P)$ s.t

$a_i = f_i(P)$ or $a\ P_i\ f_i(P)$

Theorem: (Gibbard-Satterthwaite) SCF f is SP + ONTO $\Leftrightarrow$ SP + PE iff f is dictatorial

Note: GS theorem does not hold here as pref are over objects and not alternatives. An alternative is matching of objects to agents

here it is injective (one-one) as each agent gets exactly one item and every item allocated to almost one agent

so, $A = \{a : N \to M \mid a \text{ is injective}\}$

Eg: a, a' s.t i gets n in both, then this agent is indifferent for a, a', so domain of single object allocation violates unrestricted assumptions of GS theorem

Serial dictatorship mechanism:

SCF $f: U^n \to A$ basic truthful mechanism

fixed priority (serial dictatorship)

Def: (Priority) an ordering over the agents, $\sigma : N \to N$ bijection

every agent picks her favourable from leftover list

$$a(\sigma_1) = P_{\sigma_1}(1, M)$$
$$a(\sigma_2) = P_{\sigma_2}(1, M \setminus \{a(\sigma_1)\})$$
$$\vdots$$
$$f^\sigma(P) = a$$

Theorm: f^σ is Efficient

proof: if not, then

$\exists P$ s.t $\exists a \neq f^\sigma(P)$

s.t $a(i) P_i f_i^\sigma(P)$ or $a(i) = f_i^\sigma(P)$

i.e either agent i gets same good or better $\forall i \in N$

Say first agent j in priority order of σ that has $a(j) P_j f_j^\sigma(P)$

but that $a(j)$ was avaiable when j's turn came to pick, so j cannot pick $f_j^\sigma(P)$ which is less prefered in it's turn

Theorm: f^σ is SP

proof: agents before i in order σ pick goods and i cannot influence it and i gets best ones from remaing, no reason to misreport

Modified Serial dictatorship:

$N = \{1,2,3\}$ $M = \{h_1, h_2, h_3\}$ ← h_i are houses (goods)

now priority order changes based on prefere of agent 1

$$\sigma = \begin{cases} (1,2,3) \; ; \; P_1(1) = h_1 \\ (2,1,3) \; ; \; P_1(1) \neq h_1 \end{cases}$$

Note: f^σ is SP + PE as 2, 3 cannot change priority order and so cannot manipulate

1 can change but:

Case I: $P_1(1) = h_1 \to$ most fav house

Case II: $P_1(1) \neq h_1$ then if 1 misreports to get $P_1(1) = h_1$ it will be prefed less

It is PE from same reason as normal priority dictatorship argument

8th Jan:

One sided matching:

Houses

Agents

$A \longrightarrow H_A$ endowment

P_i is total order over M (set of items)

$f(P_1, \dots P_n) = a$ → SCF

$a: N \to M$ is injective

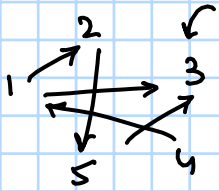
Truthfulness:

either $f_i(P_i, P_{-i})\ P_i\ f_i(P_i', P_{-i})\ \forall P_i, P_i', P_{-i}, \forall i \in N$

$f_i(P_i, P_{-i}) = f_i(P_i', P_i)$

Efficiency: $\nexists a \neq f(P)$ s.t $a_i P_i f_i(P)$ or $a_i = f_i(P)\ \forall i \in N$

Top trading cycle (TTC):



↙ every agent tells its first preference

n edges, n vertices ⇒ ∃ a cycle

↑ we remove these houses

those agents are removed

then the cycle agents are removed and process repeated

Algorithm:

Start with initial a* endowment

maintain: $M_{remaing} = M$, $N_{remaig} = N$

- Repeat, construct a directed graph where each agent in N_{remaig} points to best house in $M_{remaing}$
- Find a cycle in this graph and resolve
- update N_{remaig} and $M_{remaining}$
- stop when there is no more houses

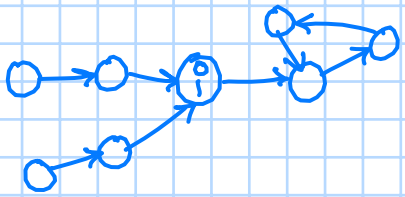
Theorm: TTC with fixed endowment is SP

proof: fix a profile P

focus on agent i: the agent gets a house under TTC (mutual) in round k when i respond P_i

case I: by misreporting P_p', the house gets in $r > K$ round then
agent i gets worse house than what was getting at K as other players have not changed pref (those will get i's first $K-1$ choices)
so agent will not misreport as not better, equal or worse only

case II: $r < K$



round r: π_{i_r} = {set of nodes who have a directed path to i}

$\pi_{i_1}, \ldots \pi_{i_K}$

Any house that agent i can get in $r < K$ is still available in round K

The house that i gets allocated in $r < K$ is less prefered than which i gets in round K

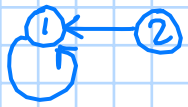
So, in both cases, the agent cannot get a better house
$\Rightarrow$ TTC is SP

Note: If TTC is not efficient (for EFF usually worse), then $\exists a'$ s.t

either $a'(i) P_p a^{TTC}(i)$ or $a'(i) = a^{TTC}(i)$

at some round $a'(i) P_p a^{TTC}(i)$ but this from definition of TTC is not possible (as this should be $\forall j \in N$)

Eg: $\sigma = (2, 1)$



then $f^{\sigma}(P) = \begin{pmatrix} 2 \to H_1 \\ 1 \to H_2 \end{pmatrix}$

$f^{TTC}(P) = \begin{pmatrix} 2 \to 2 \\ 1 \to 1 \end{pmatrix}$

Note: From above example, we see that TTC is different from SD

Stability: Group Deviation:

1	2	3	4	5	6
h_1	h_2	h_3	h_4	h_5	h_6

---

a : h_1 h_3 h_2 h_4 h_5 h_6

P_3	P_4
h_4	h_3
$\vdots$	$\vdots$
h_2	h_4

{3,4} is a "blocking" coalition of the allocation a

$\Rightarrow a$ is not a stable allocation

a^* : initial endowment

a^S : denotes the matching of the agents in S over the houses owned by the agents in S

$S \subseteq N$

Defn: (Blocking Coalition) A coalition $S \subseteq N$, $S \neq \phi$ can block a matching a at a particular pref profile P if $\exists$ a matching a^S s.t

either $a^S(i) P_i a(i)$ or $a^S(i) = a(i)$ $\forall i \in S$
and $a^S(j) P_j a(j)$, $\exists j \in S$

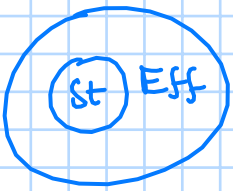
Defn: (Core) A matching is in core of profile P if no coalition can block it in P

$core(P)$, $core(P')$ ...

Defn: (Stability) An SCF f is stable if $f(P)$ is in the $core(P)$ $\forall P$

Note: stability $\Rightarrow$ Efficiency as stability is for a particular subset but Efficiency is for wide set

stronger ↑



$core(P) \subseteq A$ s.t no blocking coalition
$\forall a \in core(P)$
if $f(P) \in core(P)$ $\forall P$, f is stable

Eg: Eff $\not\Rightarrow$ St

as $\sigma = (2,1)$ is serial dictatorship
$\Rightarrow \sigma$ is Eff

but



$S = \{1\}$ blocks it to be stable

Theorem: TTC is stable

proof:

If TTC is not stable, then
$\exists P$ s.t
$\exists S \subseteq N$ coalition blocks a^{TTC} in P

$\exists a^S$ (only the houses owned by the agents in S)
s.t

$a^S(i) P_i a^{TTC}(i)$ $\forall i \in S$
or $a^S(i) = a^{TTC}(i)$
and $\exists j \in S$ s.t $a^S(j) P_j a^{TTC}(j)$

$T = \{j \in S \mid a^S(j) P_j a^{TTC}(j)\} \neq \phi$

Run TTP on P, then

the agents in S who got their house in round 1 is called S^1
$S^1 \cap T = \phi$ → as they got their best houses

round 2 agents of S : $S^2 \cap T = \phi$ as there is no better house as all best options gone for S^1, and they will all get best second pref

continuing this argument for any round

to get $S^K \cap T = \phi$

$\forall_1 \; (S^1 \cup S^2 \dots S^K) \cap T = \phi$

Theorem: ∃ a unique core matching for every P and TTC refers that

proof:

assume TTC is not unique

there

a^1 and a^{TTC}

$R1: a^1 \equiv a^{TTC}$

⋮

$RK: a^1 \neq a^{TTC}$ there agents getting allocated in round K according to TTC blocks the allocation a^1

$\Rightarrow a^1$ is not in the core

this is a contradiction as $a^1 \in core(P)$ as f is stable

$\forall_1 \; a^1 \equiv a^{TTC}$

12th Jan:

Applications of one-sided matching:

Organ donation: like liver transplant, kidney donation

live donation / deceased donation

Kidney case: 100,000 people in US suffer from kidney disease (USOS org)
in india: NOTTO (india's org)
organindia org

Kidney transplant


patient


donor

blood group, tissue type, and other components need to be matched

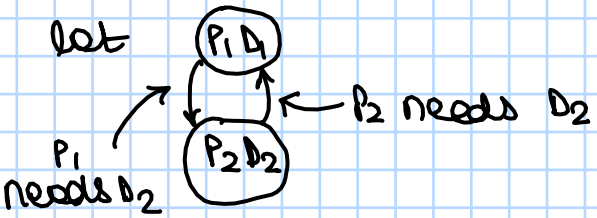
patient-donor, similar to TTC
(player-house)

first successful kidney transplant: Dr. Joseph Murray (1954, Nobel in 1990) ← He survived 8 years

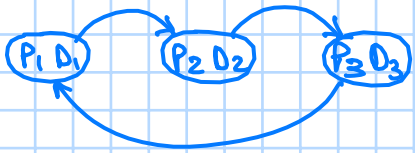
1971 - India's first - Dr. Mohan Rao (CMC Vellore)
none then 3700+ transplants in CMC

Kidney approach:

$P_1 \leftarrow D_1$
$P_2 \leftarrow D_2$



Eg:



Kidneys: Houses

preferences: Medical compatibility

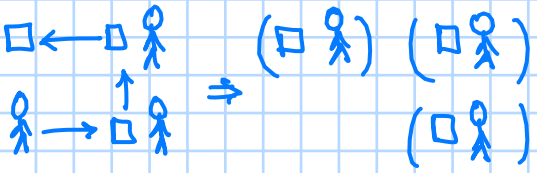
TTC in kidney exchange:

advantages: ① the TTC is in the core, no KPs can do a profitable group deviation

② every agent (patient) is weakly better off participating in this process (IR) individual rational

③ Poly-time $O(n^2)$

limitations: In practice, patient may not have donors donated kidney and opposite donor without patient ← agent without house ← Empty houses



in this case we dont need to remove cycle, we can remove some paths too

you request my house, I get your turn algorithm with numbering of players

Unbalanced market of houses:

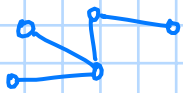
By Abdulkadiroglu and Sonmez (1999)

algorithm: you request my house, I get your turn is SP, PE

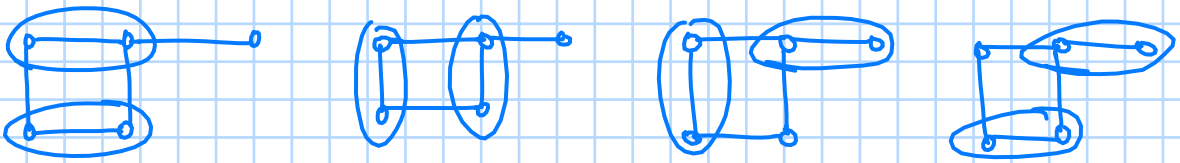
also outcome wise same as TTC

One more limitation of TTC: because of incentive issues, longer cycles are not possible
so TTC/YRMH-IGYT NOT applicable





natural thing is to save as many patients as possible
so, maximum cardinality matching



problem comes as



so max-cardinality is not good. we need some principle way of this

Priority based allocation:

fix a priority order $1,2,\ldots n$ M_0 = set of all max cardinality matching

for $i=1,2\ldots n$
let S_i be the set of matchings in M_{i-1} that matches vertex i

if $S_i \neq \phi$ then $M_i \leftarrow S_i$
else $M_i \leftarrow M_{i-1}$

Return M_n

↑ there can be multiple matchings here

NOTE: ① M_n can be set of multiple matchings

② Each matching in M_n matches same set of vertices

③ an agent who gets matched in any round is never unmatched so no agent misreports if compatible else (SPNE)

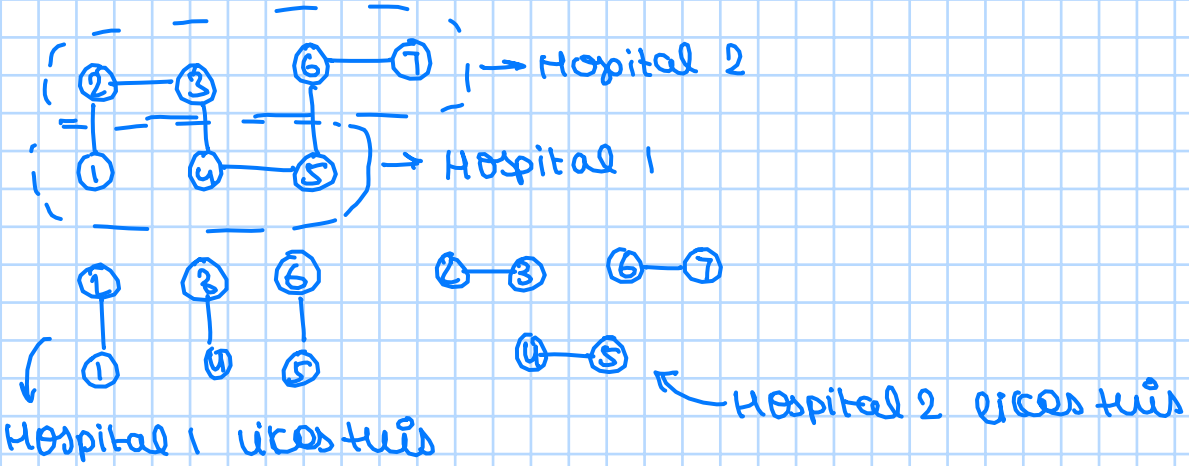
④ Pareto optimal directly from defination of max cardinality matching (PE)

⑤ polytime (Edmons)

drawback of this algorithm is that typically the data of the patients given by Hospital

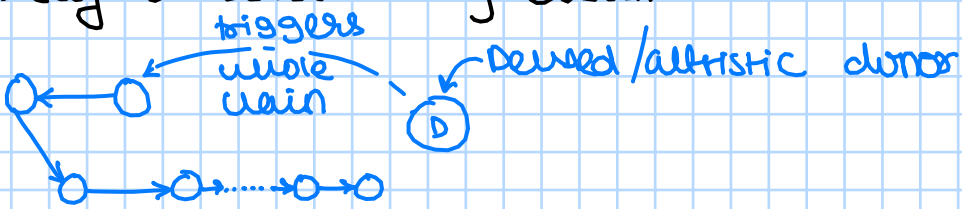
but hospital's objective is not maximum cardinality matching but how many edges lie in that Hospital

Eg:



Hospital 1 can remove (4)—(5), Hospital 1 can remove (2)—(3)

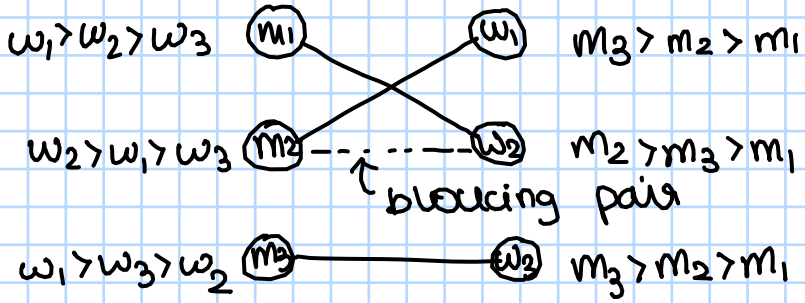
now, lets say we have a long chain



Here even if some donor backs out, the next patient will not loose its donor

15th Jan:

Two sided matching:



(m_2, w_2) is a blocking pair of the matching

Gale-Shapley Deferred Acceptance algorithm:

MP DA (men proposed differed algorithm)
WP DA (women proposed differed algorithm)

men propose women, then woose men (best from their choice) ← Round 1

round 2: men who got rejected, propose again and women accept / reject again

algorithm continious till every man has their proposal accepted, no take rejections

Note: MP ≠ WP, they can give output to diffent matchings

MP DA:

initialise all $m \in M$, $|M| = |W|$ (balanced market)
$w \in W$ as FREE

while $\exists$ m who is FREE:
m proposes to w who is the most prefered by m and has not rejected him

if w is FREE then (m, w) is a tentative match
else (m', w) matching already exist then

if $m P_w m'$ then m' is set FREE, (m, w) is tentative match

if $m' P_w m$ then m is FREE

tentative → fixed

Note: convergence is always there and we would take to modify algorithm if $|M| \neq |N|$

Note: complexity is $O(n^2)$ as if everyone has same prefere

now, pick any profile P, $\exists$ one women who gets exact one proposal

as if not then every women get's atleast 2 proposals in this case the loop (last round of algorithm) does not end, but we have shown convergence

so $(n-1)\times n + 1$ = maximum no of proposals
↑ $n-1$ women gets n proposals → cyclic case

Theorem: DA always returns a perfect matching

proof:

every man makes atmost n proposals. Every man is matched to exactly one women

every man will be matched. women get atleast one proposal, tentatively matched women is never unmatched.

Note: perfect matching is when everyone gets one partner

Pairwise stability:

matching is a bijective map $\mu : M \to N$ (N: set of women, M: set of men)

when 1-1, onto
i.e $\mu(m) = n$
then women matched to men under μ

$\mu^{-1}(w) = m$ or $\mu(w) = m$
man matched to women under μ^{-1} or μ

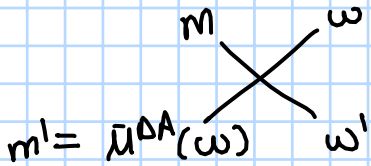
Defn: (μ is pairwise unstable) if $\exists P$, and (m, w) s.t $w P_m \mu(m)$ and $m P_w \mu^{-1}(w)$
(m, w) is a blocking pair at P

Defn: (μ is pairwise stable) if it has no blocking pairs $\forall$ preference profile
$P = (P_{m_1}, P_{m_2}, \ldots P_{m_n}, P_{w_1}, \ldots P_{w_n})$

Theorem: DA is pairwise stable

proof:

suppose not, then $\exists$ P s.t (m,w) s.t
$w P_m \mu^{DA}(m)$ and $m P_w \mu^{-DA}(w)$



but m would have proposed to w and was rejected but w must have saved a proposal from a man above m but

$m P_w \mu^{-DA}(w)$

so this is a contradiction

Note: we don't only need pairwise stability alone, we also need group blocking

Defn: (group blocking) A coalition $S \subseteq M \cup W$ blocks matching μ at a profile P if $\exists$ another matching μ'

① the coalition envoys remains within the coalition
$\forall m \in MNS,\ \mu'(m) \in WNS$
$w \in MNS,\ \mu'(w) \in WNS$

② $\forall m \in MNS,\ \mu'(m)\ P_m\ \mu(m)$
$w \in WNS,\ \mu'(w)\ P_w\ \mu(m)$

Def: (core matching) A matching μ is in the core of profile P if no coalition can block that matching

Theorem: A matching is pairwise stable at P iff it is in core of P

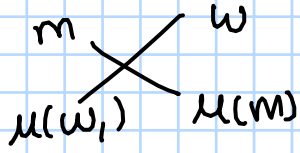
proof: ($\Leftarrow$) group stable is pairwise stable (take group to be $M \cup W$)

($\Rightarrow$) we want to show pairwise stability $\Rightarrow$ group stable then showing not group stable then not pairwise

so, $\exists$ a group coalition S where (i), (ii) from definition holds

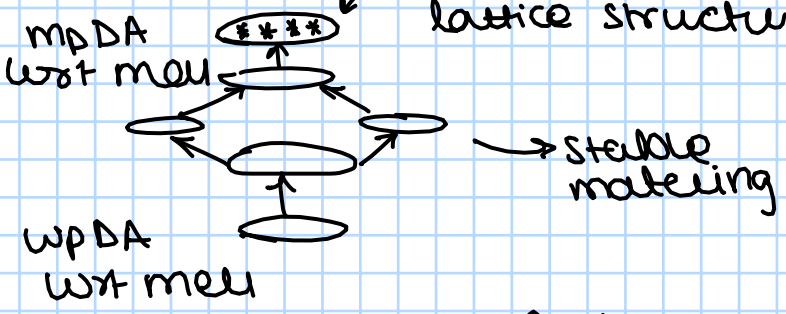
then $\mu'(m) \in S,\ \mu(w) \in S\ \forall m, w \in S$
$\mu'(m)\ P_m\ \mu(m)\ \forall m, n \in S$ (same for S)

let $\mu'(m) = w \neq \mu(m)$
$m = \mu'(w) \neq \mu(w)$



$w\ P_m\ \mu(m)$
$m\ P_w\ \mu(w)$
→ (m, w) is a blocking pair of μ in P

good stable matching:

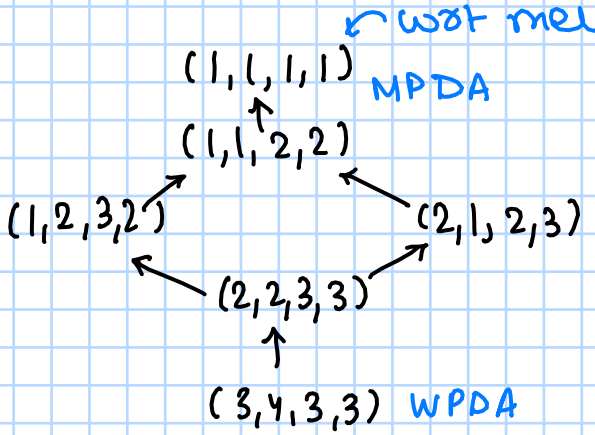


opposite to women case lattice

19th Jan :

Structure of stable matching :

$w_4 > w_1 > w_2 > w_3$	m_1	w_1	$m_2 > m_1 > m_4 > m_3$
$w_3 > w_2 > w_4 > w_1$	m_2	w_2	$m_1 > m_2 > m_3 > m_4$
$w_1 > w_2 > w_3 > w_4$	m_3	w_3	$m_3 > m_1 > m_2 > m_4$
$w_2 > w_1 > w_4 > w_3$	m_4	w_4	$m_4 > m_2 > m_1 > m_3$



lattice of stable matching

Defn: (Achievability) A man m and women w are achievable for each other if ∃ a stable matching where they are matched, for a fixed profile

Eg: for m_1, w_4, w_1, w_2 are achievable but w_3 is not as no stable matching from above diagram

The most prefered achievable women "favourite achievable" women consider a function $f^m : M \to W$, for a fixed profile

$f^m(m_1) = w_4$

$f^m(m_2) = w_3$

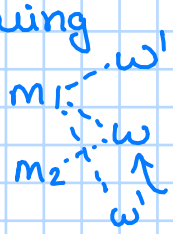
$f^m(m_3) = w_1$

$f^m(m_4) = w_2$, f^m is man-optimal matching

Exe: Is f^m a matching? As stable matchings where the favoite achievable partner is obtained may be different

Ans:

$f^m(m_1) = f^m(m_2) = w$; if this happens then f^m is not a matching



stable matching where (m_2, w) matched, then as w is favourite achievable women of m_1, w' must be above w as from defination of favoite women in stable matching

so, (m_1, w) is a blocking pair

wlog $m_1 P_w m_2$, consider stable matching μ where (m_2, w) is matched (by assumption w is allowable)

in μ (m_1, w') w' is preferred less than w
$\Rightarrow$ (m_1, w) is blocking pair

so, f^m is a matching

Theorm: for every profile P, the matching obtained by mp-DA is men-optimal

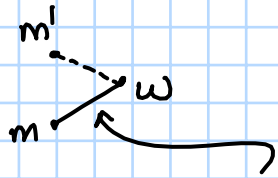
proof:

man-optimal matching exist we know, now we get that its a stable matching from this

if we show that the favourite allowable women of a man never rejects that man in a mp-DA algorithm

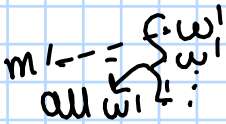
suppose not, then let m be first man in mp-DA on profile P that gets rejected by his favourable allowable women w

$f^m(m) = w$
$\Rightarrow m' P_w m$



as $f^m(m) = w$
$\exists$ matching μ (stable) where (m, w)

now in μ:



if $w' P_{m'} w$ then (any one w')
as w' was rejected m'
and w is allowable, it becomes favourite women of m'

but
w' has rejected m'
$\Rightarrow$ for allowable women rejected m'
but m is first of such case

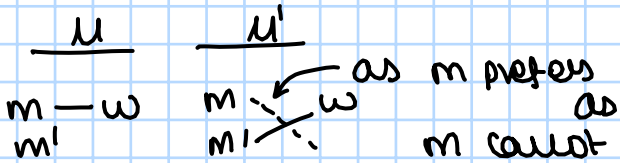
this is a contradiction

so, mp-DA gives men optimal stable match

Theorm: For any distinct stable matchings μ and μ' if all men prefer μ atleast as μ', then all women prefer μ' atleast as μ

men $\uparrow\downarrow$ women

proof: suppose all men prefer μ atleast as μ'
if $\exists$ w who strictly prefers μ to μ'

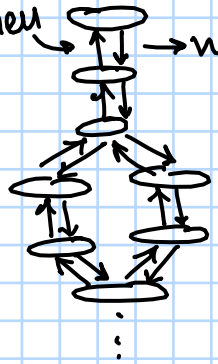


as there m will prefer μ'

$\Rightarrow m - w'$ s.t w' below w

(m, w) is a blocking pair in μ' (contradiction to stability)

so, men → women



is now lattice structure will look like

fm: favorite function points every player to his/her favorite allowable partner

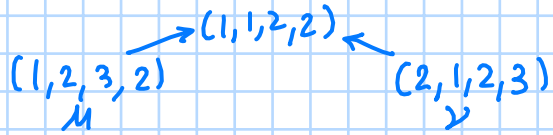
let μ, ν be two stable matchings

Defn: a mapping $max_{\mu,\nu}$ s.t

(a) each man points to his more preferred women between μ and ν

(b) each women points to her less preferred man between μ and ν

Eg:



$max_{\mu,\nu}(m_1) = p_{m_1}(1)$

$max_{\mu,\nu}(m_2) = p_{m_2}(1)$

$max_{\mu,\nu}(m_3) = p_{m_3}(2)$

$max_{\mu,\nu}(m_4) = p_{m_4}(2)$

Theorm: $max_{\mu,\nu}$ is a matching

proof: $max_{\mu,\nu}(m) = w$ iff $max_{\mu,\nu}(w) = m$ (map is bijection)

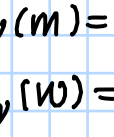
then $max_{\mu,\nu}$ is a matching (sufficient to show)

($\Rightarrow$) suppose not, $max_{\mu,\nu}(m) = w$ but $max_{\mu,\nu}(w) = m' \neq m$

μ: $m - w$

ν: $m - w'$, $m' - w$ (not possible by defn of max)

↑ less preferred



$m' P_w m$ is not possible as women points to less prefered

so, $m P_w m'$

(m, w) are blocking pair of ν (contradiction)

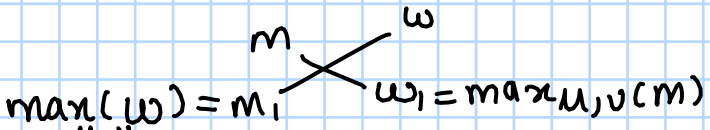
($\Leftarrow$) same argument from other side

or as $max_{\mu,\nu}$ is 1-1 and finite sets with same cardinality

$\Rightarrow max_{\mu,\nu}$ is bijective so directly other side

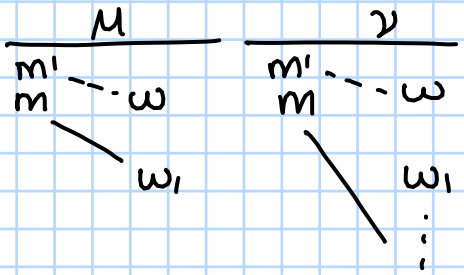
Theorm: $max_{\mu,\nu}$ is a stable matching

proof: Suppose not, then $\exists$ a blocking pair (m, w) of $max_{\mu,\nu}$



$w P_m max_{\mu,\nu}(m)$

$m P_w max_{\mu,\nu}(w)$

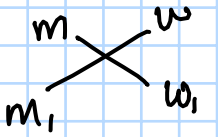


as $m P_w max_{\mu,\nu}(w)$ but if m' above for both m

then $max_{\mu,\nu}(w) \neq m'$ so, (m, w) not blocking

as $max_{\mu,\nu}(w) P_w m$

so, in atleast one of them μ / ν:



this is a contradiction

as if w is matched to a man above m in both μ, ν then $max_{m,\nu}(w)$ is more preferred than m

(m, w) cannot be a blocking pair of $max_{\mu,\nu}$

22nd Jan:

Strategy proof in Stable matching:

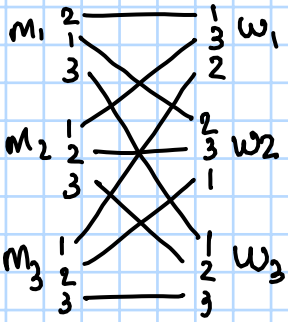
A matching μ is manipulable if $\exists P$ and a player $i \in N$ s.t

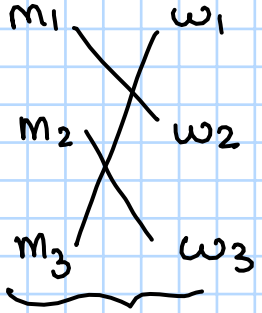
$$\mu(P_i', P_{-i}) \; P_i \; \mu(P_i, P_{-i}) \text{ for some player } P_i'$$

A matching is Strategyproof if it is not manipulable at any profile by any player

Note: DA is manipulable

$w_2 > w_1 > w_3$ notation is $\begin{matrix}2\\1\\3\end{matrix}$ or w_1 on 2nd pos, w_2 on 1st pos, w_3 on 3rd pos

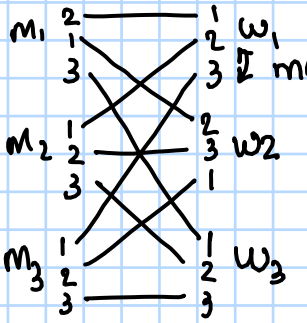




got 2nd

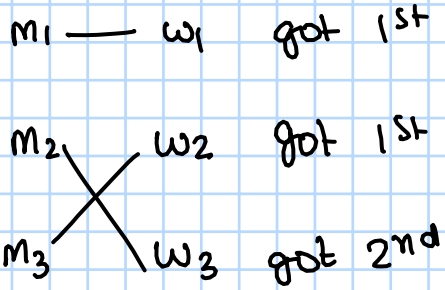
mp-DA algorithm output

by lying w_1 gets better outcome



manipulated matching

$m_1 > m_3 > m_2$
$m_1 > m_2 > m_3$

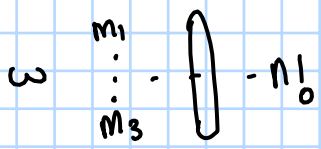


Theorem: (Dubins and Freedman) mp-DA is strategyproof for men

Now, we want to know how complex it is to manipulate

Optimal manipulation:

Defn: Among all possible manipulation a women can perform, the ones that gives her best man is the optimal manipulation w.r.t her true preferlie.



Note: optimal manipulation does not mean top of pref or best available man

Eg: $m_1 > m_2 \ldots m_8$
$m_5 > m_2 > \ldots$ → optimal manipulation

Defn: (Single elivation) Pick one man and elevate and don't change the order of the rest

Theorem: (Vaish and Garg) Any optimal manipulation for women in mp-DA can also be derived via a single manipulation

so now, complexity to optimally manipulate $= O(n^4)$

$O(n^2)$ for single elevation, $O(n^2)$ for DA

P_m P_w
m → w
proposing in original pref profile mp-DA

m w'
P_m $P_{w'} = m$ is true man, w has elivated

Theorem: If m proposed w in (P_w, P_{-w}) then m proposes w in $(P_{w'}, P_{-w})$

proof: as only one women's pref is changed, m proposes to all w' before w and gets rejected, every other man gets rejected in same manner as m is elivated (so does not change anything), then m proposes to w

Note: The Opt manipulation may not be single-elevation

Theorem: The DA algorithm after optimal manipulation is stable wrt true preference profile

proof:

P is pref profile $\quad P' = (P_{w'}, P_{-w})$

$\mu = mp\text{-}DA(P) \quad \mu' = mp\text{-}DA(P')$

we have to show μ' is stable wrt P, so if not then $\exists (m_1, w_1)$ a blocking pair in P

if $w_1 \neq w$, then μ' is not stable as w manipulated so in P'

m_1 w_1
$\mu(w_1)$ $\mu'(m_1)$ ⇒ unstable, so not possible ($\because \mu'$ is stable wrt m_1, w_1 as $w_1 \neq w$)

P'

Same as P as $w_1 \neq w$

so, $w_1 = w$ → the blocking path will be done by (m_1, w) in P

P_{m_1}	P_w	P_{m_1}	$P_{w'}$	P_{m_1}	$P_{w''}$
w	m_1				m_1
$\mu'(m_1)$	$\mu'(w)$	$w - \mu'(m)$		w	
		$\mu'(m_1) - m_1$		$\mu'(m_1)$	$\mu'(w)$

as (m_1, w) is blocking pair

moved to top as μ' is stable wrt $P' = (P'_w, P_{-w})$

property of μ'

m_1 proposes w in in P' as m_1 is matched to a woman after w and this is mp-DA

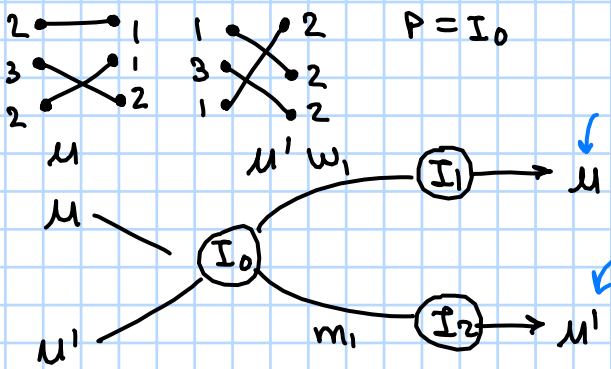
by our previous theorem, m_1 also proposes w in P''

$\Rightarrow w$ will be matched to m_1 in P''
$\Rightarrow P_{ii}$ is not optimal manipulation (as $m_1 P_w \mu'(w)$)

Note: stable DA matching is manipulable but optimally manipulated preferences are stable

Theorem: (Ruth) NO deterministic stable matching is strategyproof

proof: counterexample is our orignal example



if μ' then w_1 manipulates to get μ (only stable as mp-DA = wp-DA)

if μ then m_1 manipulates to get μ' (only stable as mp-DA = wp-DA)

$m_1 \rightarrow w_2 > w_3 > w_1$ (manipulation)

The stable marriage problem (Ref book)

29th Jan:

Fair division:

Eg: Division of spectrum resources, divison of fab is shared service (car rides), division of budget in diff hands, divison of inheritance, Hostel room allocation, essential supplies (valines)

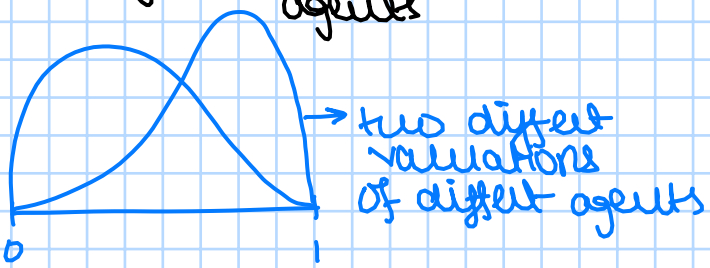
Note: Above, some divisions are divisible, ones are not (we cannot split something, indivisible)

Divisible fair divison:

common metaphor for divisible goods is cake

cake has several properties

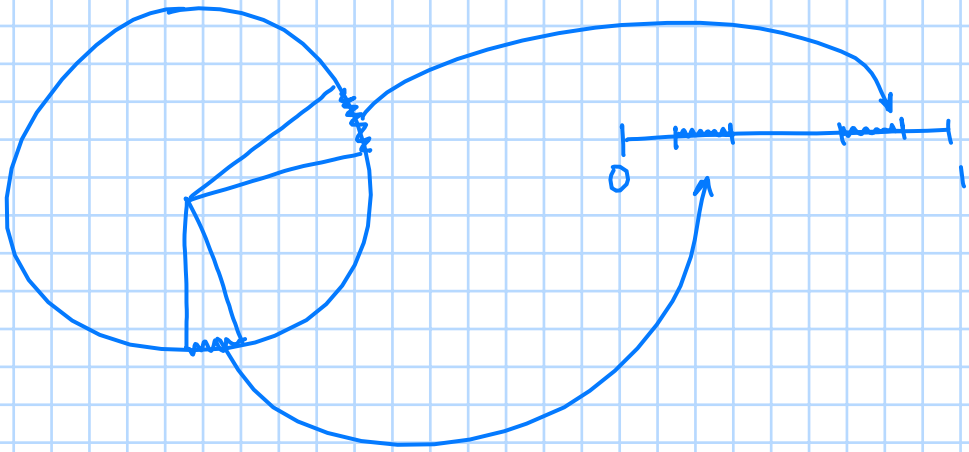
① it is heterogeneous: equal amount is valued differtly by diffet agents



② Any arbitary fractional allocation is possible, divisibility

③ non-identically valued: same piece has diffet valu for diffet agents

④ additive valued good: all valuation are non negative and valuations are additive



agent set $N = \{1, 2, \ldots n\}$

piece of a cake is a finite union of disjoint intervals

$S_i \subseteq [0,1]$

$K \leftarrow$ finite

$S_i = \bigsqcup_{k=1}^{K} I_{ik}$ → all finite

piece ↑ intervals

Agents valuation / preferces:

Presented via value fuction, $v_i : P \longrightarrow \mathbb{R}_{\geq 0}$ (P ← set of all pieces)

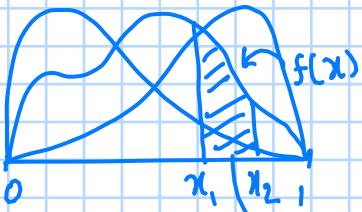
① additivity: $\forall X, Y \subseteq [0,1]$, $X \cap Y = \phi$, $v_i(X \cup Y) = v_i(X) + v_i(Y)$ $(X, Y \in P)$

② divisibility: for any $X \subseteq [0,1]$ (← cake), any $\lambda \in [0,1]$ → real interval

$\exists\ Y \subseteq X$ s.t $v_i(Y) = \lambda v_i(X)$ i.e $V_i([x,x]) = 0$

③ Normalisation: $v_i([0,1]) = 1 \quad \forall\ i \in \{1,2 \ldots n\}$

Eg:



$$V[x_1, x_2] = \int_{x_1}^{x_2} f(x)\,dx$$

Defn: (allocation) a partition $(A_1, A_2, \ldots A_n)$ A_i = Piece that goes to i

① $A_i \cap A_j = \phi \quad i \neq j$

② $\bigcup_{i \in N} A_i = [0,1]$

Fairness ideas:

① proportionality: an allocation $(A_1, A_2, \ldots A_n)$ is proportional (PROP) if $v_i(A_i) \geq \frac{1}{n} \quad \forall\ i \in N$

② Envy-freeness: an allocation $(A_1, A_2, \ldots A_n)$ is Envy-Free (EF) if $v_i(A_i) \geq v_i(A_j) \quad \forall\ i,j \in N$

Note: EF $\Rightarrow$ PROP as $v_i(A_i) \geq v_i(A_j)$

$\Rightarrow \sum v_i(A_i) \geq \sum v_i(A_j)$

$\Rightarrow n\, v_i(A_i) \geq 1$

$\Rightarrow v_i(A_i) \geq \frac{1}{n}$

2-agents:

PROP $\Rightarrow$ EF as $v_i(A_i) \geq \frac{1}{2}$

$\Rightarrow 1 - v_i(A_j) \geq \frac{1}{2}$

$\Rightarrow v_i(A_j) \leq \frac{1}{2}$

$\Rightarrow v_i(A_j) = \frac{1}{2}$

So, in $|N| = 2$, PROP $\Leftrightarrow$ EF

Eg: for 3 agents:

	A	B	C
1	$\frac{1}{3}$	$\frac{1}{3}+\varepsilon$	$\frac{1}{3}-\varepsilon$
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$v_i(A_i) = \frac{1}{3} \geq \frac{1}{3}$

Not EF as $v_1(A) < v_1(B)$

↑ 1 got ↑ 2 got

Robert-Webb Query model:

motivation comes from computation of qualifying the computation of cake-cutting

we ask a question: $Cut_i(x,\alpha) \to$ start at x and give a piece of α

$Cut_i(x,\alpha)$ = gives the point K where $v_i([x,K]) = \alpha$

one more question: $eval_i(x,y) = v_i([x,y])$

cut and choose:

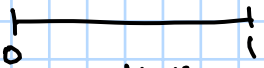
PROP: as cutter cuts in half, picker will choose bigger piece

$Cut_1(0,\frac{1}{2}) = \alpha$ ↑ ↳ $Eval_2(0,\alpha) < \frac{1}{2}$ pick other one

else pick $[0,\alpha]$

so, 2 RW queries

moving knife-protocol:



→ Knife moves, first one to get buzzer $\frac{1}{n}$ we get that piece to that person

last agent $\geq \frac{1}{n}$ as did not press buzzer

algorithm:

Repeat:
- $x = 0$, $Rem = N$
- $\forall i \in Rem$, $cut_i(x, \frac{1}{n}) = K_i$
- $i = \arg\min_i K_i$, give $[x, K_i]$ to i^{th} player
- $Rem = Rem \setminus \{i\}$
- Break if $|Rem| = 1$ and give rem to the last player

every round = $|Rem|$ cut queries

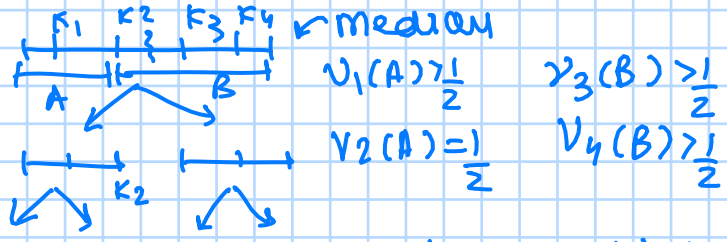
$\Rightarrow$ total $n, n-1, \ldots 1$

or $O(n^2)$ in RW queries

also called as last protocol (I cuts, then if someone wants make it smaller or let i take)

we want to know if for PROP, $O(n^2)$ is best we can do

eg: $|N| = 2^k$



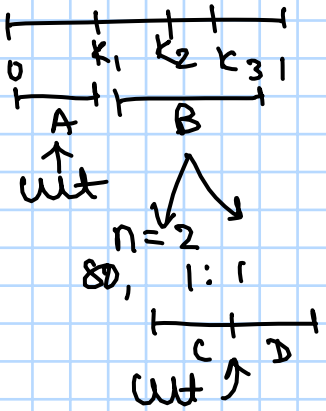
once again to get all $\geq \frac{1}{4}$

(in this case $O(n \log n)$)

in general n case: every agent gives $\lfloor \frac{n}{2} \rfloor : \lceil \frac{n}{2} \rceil$ and first $\lfloor \frac{n}{2} \rfloor$

agents take away first part
rest second part

$n=3$: 1:2 $\left\lfloor \frac{n}{2} \right\rfloor = 1$



$n=2$

so, 1:1

$O(n \log n)$ is better protocol

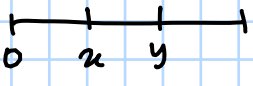
2nd Feb:

Fair division:

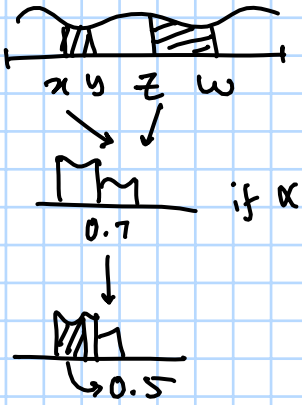
We saw PROP $v_i(A_i) \geq \frac{1}{n} \; \forall i \in N$

ENVF $v_i(A_i) \geq v_i(A_j) \; \forall i,j \in N$

$A = (A_1 \ldots A_n)$ is an allocation, we also saw the RW-Query model



$cut(x, \alpha) = y$



we can apply cut like this also by putting pieces side by side

other query is $eval([x,y]) = v_i([x,y])$
if we have a piece (finite union) is same as valuation is additive

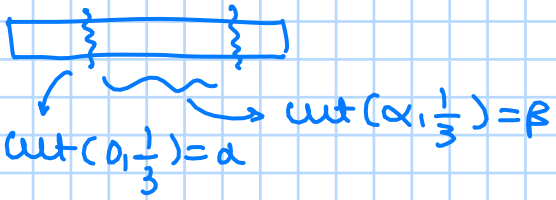
best algorithm for PROP is $O(n \log n)$ (Even Paz protocol)

Envy freeness:

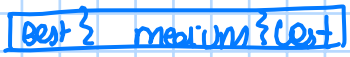
3 elements:
we want $v_i(A_i) \geq v_i(A_j) \; \forall i,j$ this will imply prop



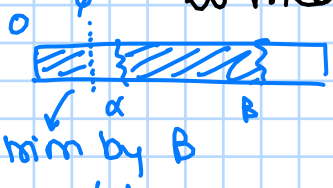
A, B, C agents, A cuts 3 equal pieces acc to v_A



now B has to order all these pieces s.t Best, medium, least



and let B trim best piece s.t it is equally valued by B to medium piece



now C will pick the favorite piece then B then A, A cannot have trimmed piece the Best piece by B trimmed

if C doesn't pick the trimmed piece, B gets it

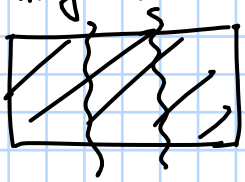
upto this its envyfree as A is envyfree, B is envyfree, C is envyfree
this is trivial

now we have give the original trimming left, let T = agent who got trimmed piece (Best by B)

T = B or C

$T' = \{B, C\} \setminus T$

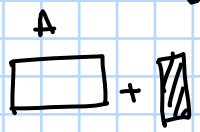
T' to cut the trimmings into 3 equal pieces into $V_{T'}$



A was dominant i.e trimmings to go anyone, doesn't matter

A will never be envyous of T as A got $\frac{1}{3}$

So, T selects first, then A then T' gets last



make valuation then

if T = B, then B gets Best + piece
T = C, then B gets BEST

same for T'

Aziz-Mckenzie:

the n-player has $O(n^{n^{n^{n^{n}}}})$ n ton.. 6 times

lower bound is $\Omega(n^2)$



local envyfreeness property:

we only care about our local neighbours

EF ⇒ PROP

$V_i(A_i) \geq V_i(A_j)$ iff i, j are neighbours → LEF

also, $V_i(A_i) \geq \frac{1}{|N_i|} \sum_{K \in N_i} V_i(A_K)$ → locally proportional → LPROP

(N_i: set of neighbours; $\sum$: sum)

EF ⇒ LEF ⇒ LPROP

⇓

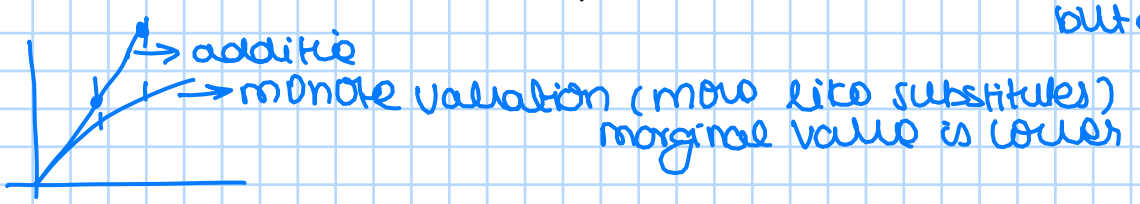
PROP

Indivisible item allocation:

Additive valuation: $v_i(\{a,b,c\}) = v_i(a) + v_i(b) + v_i(c)$

monotone valuation: $v_i(T) \geq v_i(S)$ if $S \subseteq T$

} Both are different but additive ⇒ monotone



Eg:

	A	B	C	D	E
1	1	2	5	1	1
2	1	0	5	1	1
3	1	1	5	1	1

$N = \{1, \ldots n\}$, $M = \{1, \ldots m\}$

$A_i \cap A_j = \phi$, $A_i \subseteq M$

$\bigcup_{i \in N} A_i = M$

$v_i : 2^M \to \mathbb{R}_{\geq 0}$

↳ useless gets 5, others will be EF

EF:

$v_i(A_i) \geq v_i(A_j) \quad \forall i,j \in N$

	A	B	C
1	2	1	2
2	1	1	3

$A_1 = \{A, B\}$ $A_2 = \{C\}$

$v_1(A_1) \geq v_1(A_2)$

$v_2(A_2) \geq v_2(A_1)$

Note: From above example, EF not always guaranteed in divisible allocation

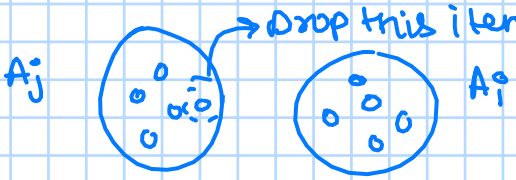
Note: checking whether EF allocation exist is NP-complete (Aziz etal)
(for only indivisible items, as divisible is always there)

we want to relax EF

EF upto one good:

An allocation, $A = (A_1, A_2, \ldots A_n)$ is EF-1 if $\forall i,j$

$$v_i(A_i) \geq v_i(A_j \setminus \{a_j\}) \text{ for some } a_j \in A_j$$



$v_i(A_j) > v_i(A_i)$

but

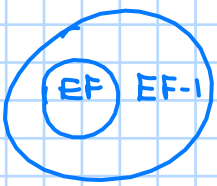
$v_i(A_i) \geq v_i(A_j \setminus \{a_j\})$

Note: this a_j depends on i i.e $a_{j,i}$

we also have EFX, every free upto any group

$$v_i(A_i) \geq v_i(A_j \setminus \{a_i\}) \quad \forall a_i \in A_j$$

$EF \Rightarrow EFX \Rightarrow EF\text{-}1$



Round-Robin algorithm:

$1 \to 2 \to 3 \to \dots \to n$ (picking order) $m \gg n$ let $m = kn + r$

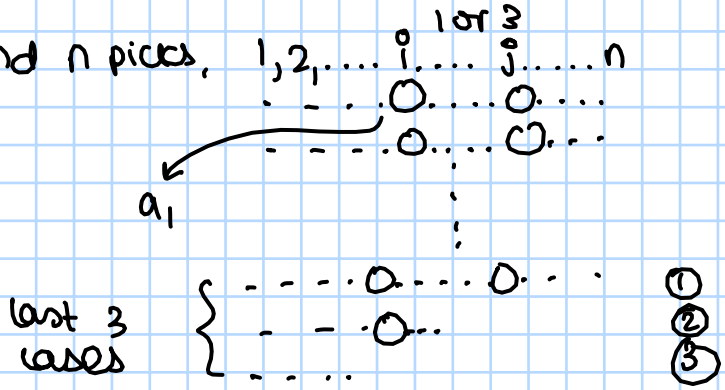
Each will pick their fav one item from available items

Theorem: RR is EF-1

proof: 2 items, 2 agents trivial

3 items, 2 agents $1 \to ① ③$, $2 \to ② \to$ Best except best is 1 or 3

let's say every round n picks, $1, 2, \dots i \dots j \dots n$

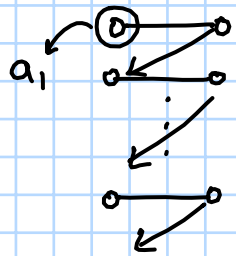


$i < j$ then i picks first

$v_i(A_i) \geq v_i(A_j)$ or i is EF j

$\Rightarrow i$ is EF-1

now, in all 3 cases



first picked by i

$v_j(A_j) \geq v_j(A_i \setminus \{a_1\})$

this happens as valuations are additive

we will now see how to extend this to monotone class

9th Feb:

Envy-free upto one good for indivisible items

- For indivisible items, EF allocation may not exist
- valuations are additive - EF1 always exist (∵ Round robin algorithm)

$v_i(\{a_1, a_2\}) = v_i(a_1) + v_i(a_2)$

- monotone valuation:

$v_i(S) \leq v_i(T) \quad \forall\ S \subseteq T, \forall i$

Note: Not for all monotone valuations, round work

RR not EF1 for MONOTONE valuations:

	A	B	C	D	E	{B,D}	{A,C}	{A,E}	{C,E}
1	[10]	8	[6]	5	[1]	14	16	15	15
2	10	[8]	6	[5]	1	14	16	15	15

↳ the round-robin allocation

Final allocation: $1 \to \{A, C, E\}$

$2 \to \{B, D\}$

this is not EF_1 as $v_2(\{B,D\}) = 14$

$< v_2(\{A,C,E\} \setminus a_i)$

$\forall a_i \in \{A,C,E\}$

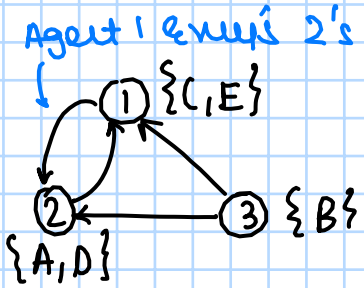
$\therefore$, v_2 is not EF_1

Envy-cycle Elimination:

Given any allocation, construct its envy-graph

	A	B	C	D	E
1	4	2	[5]	4	[1]
2	[1]	0	5	[1]	1
3	1	[1]	5	1	1

↳ some allocation



now we will try to do elimination of envy-cycle ($O(n^2)$ per item is worse as $O(n^2)$ cycles)

order the items arbitrarily, while trying to allocate item, construct the envy graph with the items allocated so far ↳ $O(mn^2)$ by below lemma

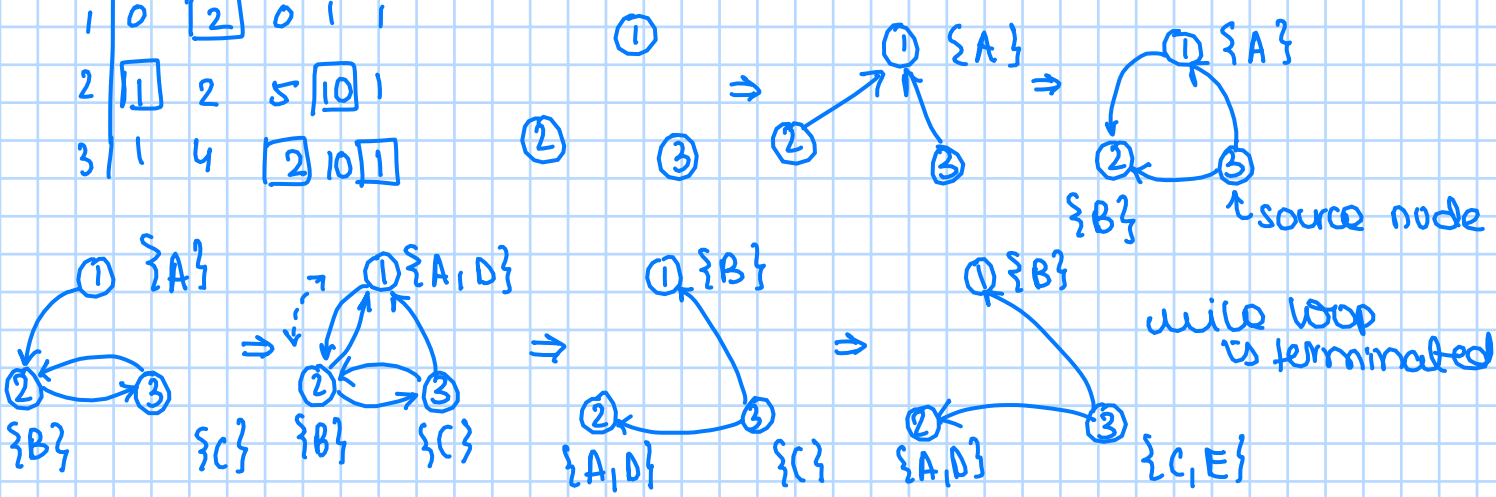
if the graph has source vertex, then assign an item to that vertex

if there is no source vertex, resolve envy cycles until a source is achieved ← give bundle to agent pointing bundle in cycle

Eg:

order: A > B > C > D > E

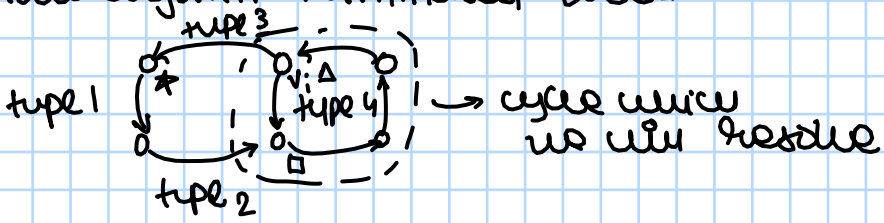
	A	B	C	D	E
1	0	**2**	0	1	1
2	**1**	2	5	**10**	1
3	1	4	**2**	10	**1**



final allocation is EF-1

Note: Above algorithm ① terminates
② is poly-time
③ gives EF, algorithm

If this algorithm terminates, then



Lemma: After resolving any cycle, the no of edges in envy graph decreases strictly

proof:

type 1 → unaffected

type 2 → unaffected

type 3 → valuation of that agent (v_i) valuation goes up

$v_i(D) > v_i(\Delta)$

$v_i(\star) > v_i(\Delta)$

now, $v_i(D) \geq v_i(\Delta)$ or $v_i(\star) > v_i(D)$
so, type 3 may decrease (not strictly)

type 4: strictly decrease, goes down by 2 atleast
(∵ cycle min length 2)

so, no of edges decreases

Lemma: Envy-free upto one good for indivisible items

proof: case 1: when allocating a new item:
when there is a source node

this is trivial as prev was EF-1 (Induction Hypothesis)

Case 2: only edges that can happen is with type 3/4:

$V_i(A_i) \geq V_i(A_j \setminus \{a_j\})$

$V_i(A_i') \geq V_i(A_i) \geq V_i(A_j \setminus \{a_j\})$ → valuation weakly improved

so, for all edges this is still EF-1

Theorem: ECE terminates in poly-time and is EF$_1$

proof: in every stage partial allocation is EF, and from lemma of deleting edges, we get poly-time ← $O(n^2)$, $O(mn^2)$ above

Exe: Show that $\exists$ EF$_1$ allocation not generated by EE or ECE, i.e they don't maintain EF$_1$ invariant + partial allocation → done down

Empty allocation is EF (not possible by defn of allocation)

Complete allocation

	A	B	C	D	E
1	4	3	1	1	1
2	5	2	1	1	1

← Round Robin not PO

this allocation is
① complete
② EF$_1$ (∵ RR)
③ not PO

Defn: (pareto optimality) An allocation A is PO if $\nexists$ B s.t $V_i(B_i) \geq V_i(A_i)\ \forall i \in N$ and $\exists j \in N$ s.t $V_j(B_j) > V_j(A_j)$

above not PO as: $1 \to \{A, B\}$, $2 \to \{C, D, E\}$

	A	B	C
1	4	3	1
2	5	2	1

(∵ EF cycle)

	A	B	C
1	4	3	1
2	5	2	1

→ not PO

← EFCE not PO

so, every cycle allocation also not PO

Note: RR, ECE not PO

now, EF$_1$ + PO items very complex to satisfy together for indivisible

Exe: Show that $\exists$ EF$_1$ allocation not generated by EE or ECE, i.e they don't maintain EF$_1$ invariant + partial allocation

Ans:

	A	B	C
1	1	2	3
2	3	2	1

we make sure we start with all items allocated to one agent

① $\{A, B, C\}$ then we will let ② pick items then ① ....

② both EF$_1$

12th Feb:

Indivisible good allocation:

EF1 : $v_i(A_i) \geq v_i(A_j \setminus \{a_j\})$ $\exists a_j \in A_j$ $\forall i,j$

PO : $\nexists B$ s.t $v_i(B_i) \geq v_i(A_i)\ \forall i \in N$
$v_i(B_i) > v_i(A_i)\ \exists i \in N$

RR, EGE $\Rightarrow$ xPO
↳ EF1 for monotone
↳ EF1 for additive

Eg:

	A	B	C	D	E
1	20	16	12	8	4
2	16	8	7	6	4

B → A (Pareto improvement)
36 40
14 16

$v_2(A_1) = 25 > 16 = v_2(A_2)$

$v_2(A_1 \setminus \{x\}) \geq 17 > 16 = v_2(A_2)$ not EF1

∴, even a pareto optimal allocation did not give EF1

we want to allocate PO + EF1, it is possible on additive valuations

Note: above EF1 + PO is attained with nash social welfare

$$A = (A_1, A_2, \ldots, A_n)$$
$$v_1(A_1) \times v_2(A_2) \times \cdots \times v_n(A_n)$$

Eg: 10 (seller) x 20 (buyer) 2 player bargain

$(x-10)$ × $(20-x)$ = Nash product

seller utility, buyer utility

when we maximise product, we get

	A	B	C
1	4	3	2
2	5	2	2

NSW(O) = 4×4 = 16
NSW(□) = 5×5 = 25

pick $A^* \in \arg\max_A NSW(A)$

Nash optimal allocation is the one that maximises NSW (if the optimal is zero then THE LARGEST SET OF AGENTS who can simultaneously value positive value, and then maximise nash product)

theorem: Nash optimal allocation is EF1 and PO for additive valuations

proof: if any allocation is not PO, then its nash social welfare can strictly improved (∵ trivial)

∴, not PO $\Rightarrow$ not NO
NO $\Rightarrow$ PO

now, if it is not EF1 then $\exists i, j$ s.t $\forall a_j \in A_j$ $v_i(A_i) < v_i(A_j \setminus \{a_j\})$

$(\square, \ldots \underset{A_j^0}{\square})$ and $(\underset{g_1}{\square}, \underset{g_2}{\square}, \ldots \underset{A_j}{\square})$

$$g^* \in \underset{\substack{g \in A_j^0 \\ v_i(\{g\}) > 0}}{\operatorname{argmin}} \frac{v_j(\{g\})}{v_i(\{g\})} \quad g \in A_j^0$$

$\exists g \in A_j$ s.t. $v_i(g) > 0$ as if all $= 0$ then $v_i(A_j^0) = 0$, so g^* never exist

now, let $(\underset{\tilde{A}_p^0}{\square}, \square \ldots \square, \underset{g^*}{\square})$ $(\underset{\tilde{A}_j}{\square}, \ldots \square) \setminus \underset{g^*}{\square}$

$\tilde{A}_p^0 = A_p^0 \cup \{g^*\}$ and $\tilde{A}_j = A_j \setminus \{g^*\}$

let $B = (A_1, \ldots \tilde{A}_p^0, \ldots \tilde{A}_j \ldots A_n)$

if $NSW(B) > NSW(A)$, then we are done

so all we have to show is

$$[v_i(A_i) + v_i(g^*)][v_j(A_j) - v_j(g^*)] > [v_i(A_i)][v_j(A_j)]$$

$$\Rightarrow \cancel{v_i(A_i) v_j(A_j)} + v_i(g^*) v_j(A_j) - v_i(A_i^0) v_j(g^*) - v_i(g^*) v_j(g^*) > \cancel{v_i(A_i) v_j(A_j)}$$

$$\Rightarrow v_j(A_j) > \frac{v_j(g^*)}{v_i(g^*)} \left[v_i(A_i^0) + v_i(g^*) \right]$$

now, $\frac{v_j(g^*)}{v_i(g^*)} \leq \frac{v_j(g)}{v_i(g)} \quad \forall g \in A_j$ ← this comes from $\underset{g \in A_j}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$\frac{a}{b} \leq \frac{c}{d}, \frac{e}{f} \Rightarrow \frac{a}{b} \leq \frac{c+e}{d+f}$

$$\Rightarrow \frac{v_j(g^*)}{v_i(g^*)} \leq \frac{\sum v_j(g)}{\sum v_i(g)} = \frac{v_j(A_j)}{v_i(A_j)}$$

$$\Rightarrow v_j(A_j) \geq \frac{v_j(g^*) v_i(A_j)}{v_i(g^*)}$$

and as $v_i(A_i) < v_i(A_j \setminus \{g_j\})$

$$\Rightarrow v_i(A_i) + v_i(g^*) < v_i(A_j)$$

$$\Rightarrow v_i(A_j) > \frac{v_i(g^*)}{v_i(g^*)} [v_i(A_i) + v_i(g^*)]$$ ← this comes from not EF_1

$\Rightarrow$ Not NO

theorem: maximising NSW is APX-complete

now, EP$_1$ + PO allocation can be allowed via a pseudopolynomial time algorithm (additive)

$O(poly(m, n, V_{max}))$ $\quad V_{max} = \max_{i,j} v_{i,j}$

this is pseudo-polytime as running time depends on valuations

and for bounded valuation, this gives poly-time algorithm

0.69 approximation of NSW

Items that reduce the valuation:

goods: items g is a good if:
$v_i(S \cup \{g\}) - v_i(S) \geq 0 \quad \forall i, \forall S \subseteq M$

bads/chores: items b is a bad if:
$v_i(S \cup \{b\}) - v_i(S) \leq 0 \quad \forall i, \forall S \subseteq M$

Eg: cloud, spectrum, vaccines → goods
corrosive assignments → bads

EF is not possible even in chores setting

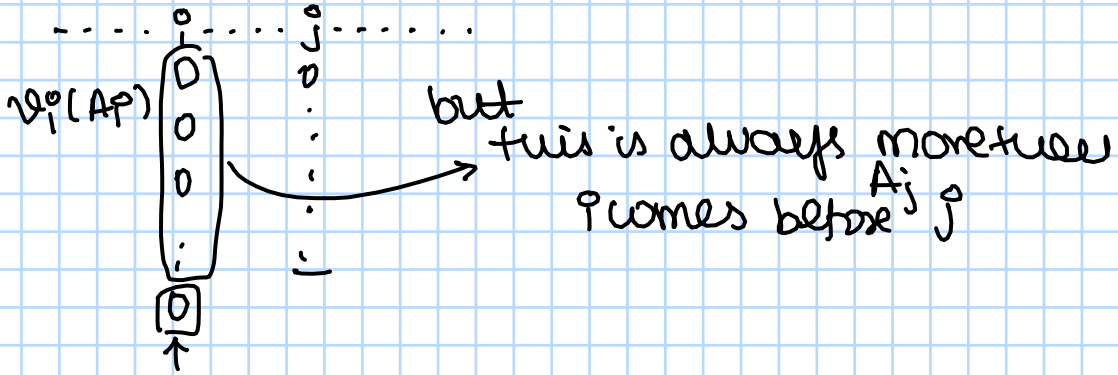
EF upto one item/EF$_1$: $v_i(A_i \setminus \{a_j\}) \geq v_i(A_j) \quad \exists a_j \in A_j \; \forall i, j$

this definition,

good: $v_i(S) \leq v_i(T)$ $S \subseteq T$ $\forall i \in N$
bad: $v_i(S) \geq v_i(T)$ $S \subseteq T$ $\forall i \in N$

↓ equivalent definition for good, same for bad

we want to know if RR algorithm works for EF$_1$ for items that are chores/bads (additive)



for j:



⇒ RR is EF1 for chores

16th Feb:

Indivisible chore allocation:

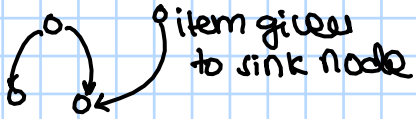
Target: EF_1: $v_i(A_i \setminus \{a_i\}) \geq v_i(A_j) \quad \forall i,j$

Case I: Additive valuation: we saw RR gives EF_1

Case II: monotone valuation: $v_i(T) \leq v_i(S) \quad S \subseteq T$

let's see if ECE algorithm works for monotone valuation or not

let's order items, then introduce sink node (instead of source)

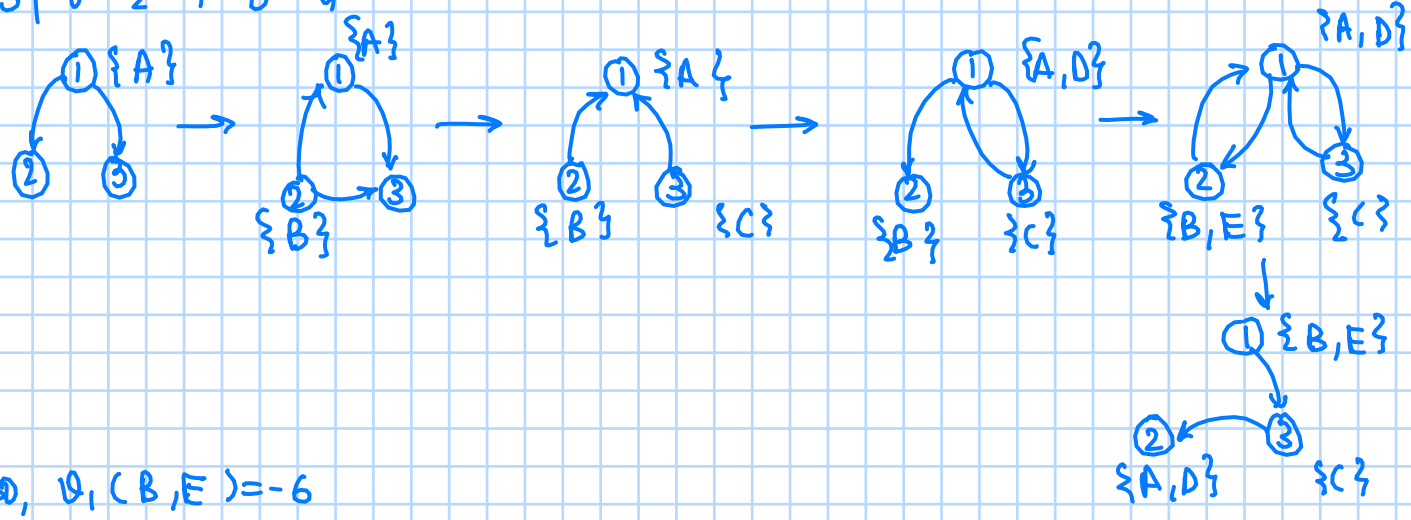


algorithm: while unallocated item:
- allocate it to a sink node of every graph

if no sink node then $\exists$ cycle
$\Rightarrow$ resolve cycle until sink node

Eg:

	A	B	C	D	E
1	-1	-3	-2	-9	-3
2	0	-1	-4	-1	-2
3	0	-2	-1	0	-4



So, $v_1(B,E) = -6$

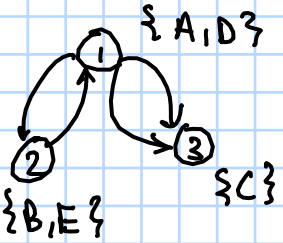
$v_1(C) = -2$ $\quad v_1(\{B,E\} \setminus \{E\}) = -3 < v_1(C)$

$v_1(\{B,E\} \setminus \{B\}) = -3 < v_1(C)$ } this is not EF_1

Note: As we are dropping something from our own allocation, we don't ensure EF_1 at every step as after cycle resolution, we got 2 items no matter what we drop it is less than what we want

So this will not work, we need, we need something else

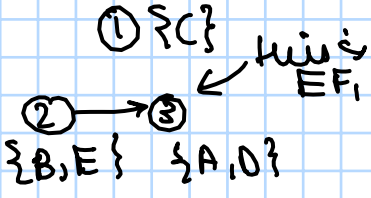
Quantum of envy:



$v_1(A,D) - v_1(B,E) = 6$

$v_1(A,D) - v_1(C) = 8$

if we resolve other cycle then

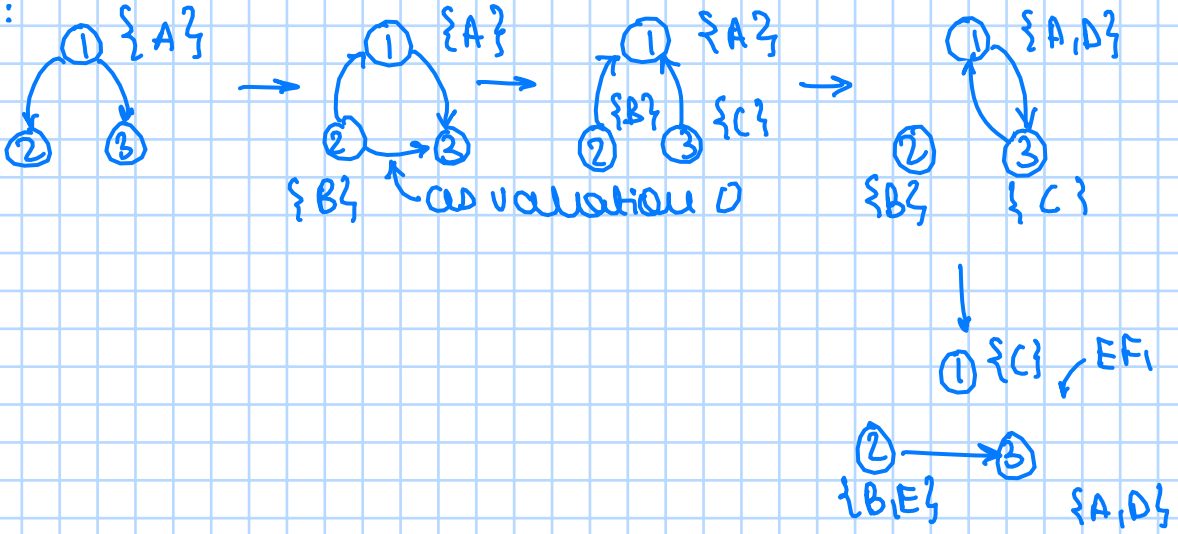


to us will every look at top ranked bundle (similar to house allocation)

i.e every agent points to their most fav bundle forcing the max-every graph

Note: If both same, then both edges together

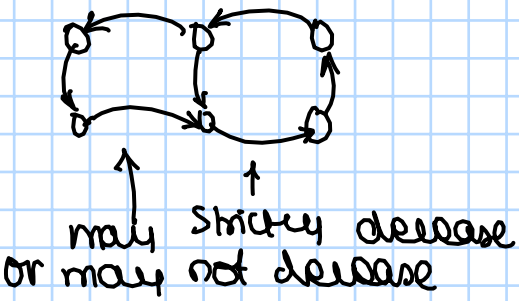
Eg:



theorem: top trading ECE converges

proof:

after resolving a max-every cycle the no. of edges strictly decrease



theorem: top-trading ECE is EF1

proof:

max every ensures that the bundle after resolving the cycle makes all agents in cycle every-free (by induction)

for all other agents, EF1 condition remains unaffected

mixed agents:

good for some agent, bad for others

	A	B	C	
	A_1			
1	3	-1	-1	A_1
2	-4	1	-2	A_2
	A_2			

for $A = (A_1, A_2)$ not EF

$v_1(A_1) = -1$

$v_1(A_2) = 2$

but $\tilde{A} = (\tilde{A_1}, \tilde{A_2})$ is EF

Lem: (Every free upto one item) an allocation $A = (A_1, A_2, \ldots A_n)$ is EF-1

if $\exists x \in A_i \cup A_j$ s.t $v_i(A_i \setminus \{x\}) > v_i(A_j \setminus \{x\})$

as $x \in A_i$ or $x \in A_j$ as $x \notin A_i \cap A_j = \phi$

Note: usual RR does not work here

	A	B
1	(1)	-1
2	1	(-1)

→ allocation not EF$_1$

new algorithm:

	A	B	C	D	E
1	-4	-1	-2	2	-4
2	0	-1	-5	-2	-1
3	-4	-2	-5	0	2

Negative items | positive items

positive: valued positive for atleast one item

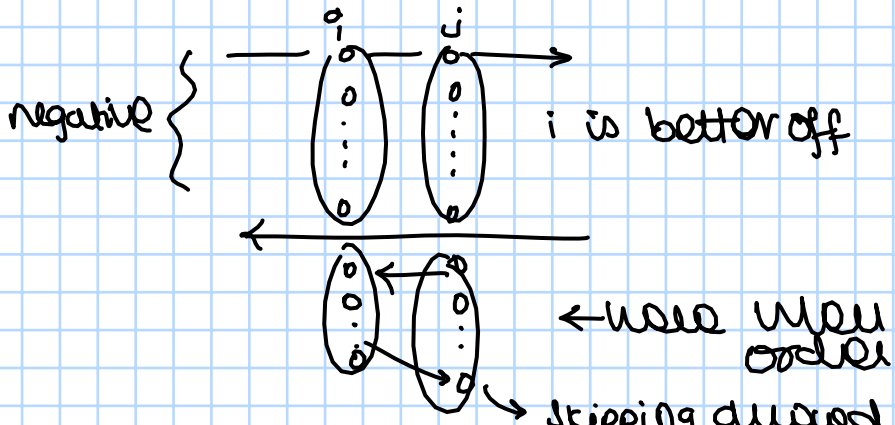
negative: valued so $\forall$ agents

doubly-RR:

phase 1: Allocate all negative items

- if items not a multiple of n then add dummy items with value=0 for all agents
- Run RR with order $1, 2, \ldots n$

phase 2: Round Robin with reverse order



Note: Doubly-RR for additive mixed is EF$_1$ (proof is similar to before)

Note: everyone comes and skip could happen as $\exists$ agent s.t good is positive

from above 2 pts we see doubly-RR also converges and converges to EF$_1$

EF$_1$ + PO:

this is going to be skipped, in case of goods: RR did not

monotone extension:

it is called doubly-monotone, each agent can mark an item as good or bad, $v_i(S \cup \{x\}) - v_i(S) \geq 0$ $\forall S$ and strict for atleast one S

then x is called good

bad: $\forall S, \nu_i(S \cup \{x\}) - \nu_i(S) < 0$ (no strictness required)

then x is bad

ECE + top trading ECE:

positive: at least one agent marks it as a good

negative: everything else

phase-1: allocate positive items using ECE (only among the agents who like them goods)

phase-2: Allocate negative using TT-ECE algorithm

as EEC on goods will produce EF1, and then we will reverse order so it will stay EF_1 like in roundrobin

then by induction we will be done

so, we have seen:

		EF1:	PO:
goods	additive	RR	NSW → EF_1 + PO
	monotone	ECE	
chores	additive	RR	unknown
	monotone	TT-ECE	
mixed	additive	D-RR	
	monotone	ECE + TT-ECE	

19th Feb:

Bargaining games:

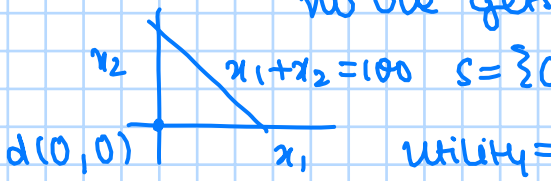
Uses of cooperative games:

Bargaining is the first model

Setting: A set of possible outcomes are bargained on, and finally certain outcomes are recommended to the players by an arbitrator

If no agreement is reached, a disagreement outcomes occur

Eg: 2 players → divide 100£, then bargain on the division, no agreement no one gets anything

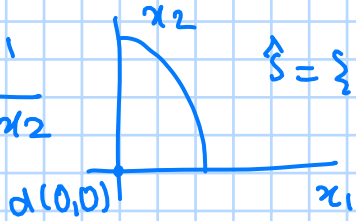


$S=\{(x,100-x) \mid x\in[0,100]\}$

utility = amount paid

$u_1(x_1)=x_1$

$u_2(x_2)=\sqrt{x_2}$



$\hat{S}=\{(x,\sqrt{100-x}) \mid x\in[0,100]\}$

Model:

A set S: set of all feasible allocations, a disagreement point $d\in S$

$$S\subseteq \mathbb{R}^2$$

Bargaining problem: (S,d)

Notation: $x,y\in\mathbb{R}^2$ $x,y\in\mathbb{R}^2$

$x\geq y \Rightarrow x_i\geq y_i \ \forall i,\ x_j^\circ > y_j^\circ \ \exists j^\circ$

$x>>y \Rightarrow x_i^\circ > y_i^\circ \ \forall i^\circ$

$aS=\{(a_1x_1,a_2x_2) \mid x\in S\}$

↑ a is a vector

$a+S=\{a+x \mid x\in S\}$

Bargaining games:

(S,d) problem, $\exists x\in S$ s.t $x>>d$

outcome: selecting a point in S

solution concept: in bargaig game ϕ s.t $\phi(S,d)\in S$

Assumption on S:

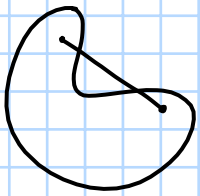
① S is compact, $S\subseteq\mathbb{R}^2$ (i.e S is closed and bounded)

↑ all limit points of S inside

↑ $\exists$ a ball of finite radius

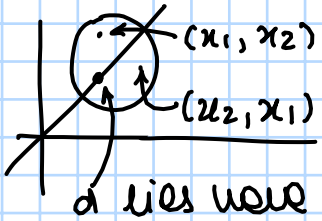
② convex: every chord of S lies within S completely

$\forall x, y \in S \quad \lambda x + (1-\lambda) y \in S \;, \lambda \in [0,1]$



Desirable properties:

① Symmetry is "equal treatment of equals"



symmetric bargaing game:

$(S, d) \quad d = (d_1, d_2)$

① $d_1 = d_2$

② $\forall (x_1, x_2) \in S \Rightarrow (x_2, x_1) \in S$

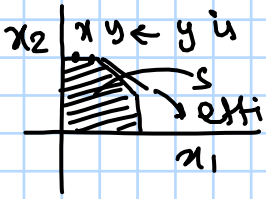
$\phi_1(S,d)$ → player 1

$\phi_2(S,d)$ → player 2

for every SBG (symmetric bargaining game)

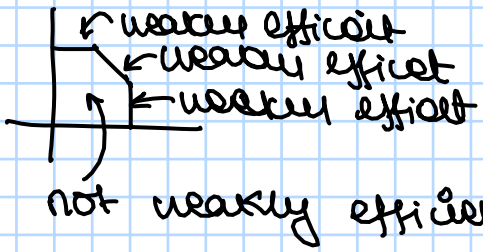
$\phi_1(S,d) = \phi_2(S,d)$

② Efficiency: an alternative $x \in S$ is efficient if $\nexists y \in S$ s.t $y \gg x$ (all agents weakly prefer y and one strictly prefers it)



$d = (0,0)$

now, an alternative $z \in S$ is weakly efficient if $\nexists y \in S$ s.t $y \gg z$

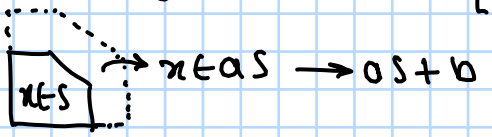


$PO(S)$ = set of all efficient points of S

Note: A solution concept ϕ is efficient if $\forall (S,d) \quad \phi(S,d) \in PO(S)$

③ covariant under positive affine transformation

① Scaling: $a \in \mathbb{R}^2_{>0} \quad a = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

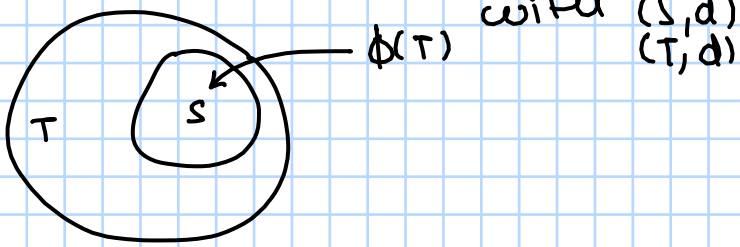


② translation: $ax + b \quad , b \in \mathbb{R}^2$

$aS + b = \{ax + b \mid x \in S\}$

if $\phi(\tilde{S}) = \phi(as+b)$
$= a\phi(S) + b \quad \forall S$
then solution concept ϕ satisfies CPAT

④ Independence of irrelevant alternatives (IIA)

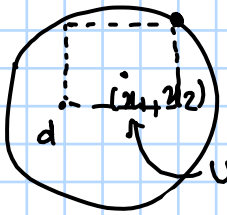


with (S,d) (T,d)

A solution concept ϕ satisfies IIA if
$\forall$ bargaing game (T,d) and every $S \subseteq T$

$$\phi(T,d) \in S \Rightarrow \phi(S,d) = \phi(T,d)$$

Nash solution (Bargaining solution)



$$\underset{(x_1,x_2)\in S}{\operatorname{argmax}} (x_1-d_1)(x_2-d_2) = N(S,d)$$

we try to fit a rectangle with max area

Lemma: $\exists$ a unique solution concept N for the family of bargaining games satisfying

EFF, SYM, IIA, CPAT where

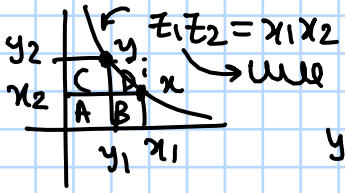
$$N(S,d) = \underset{(x_1,x_2)\in S}{\operatorname{argmax}} (x_1-d_1)(x_2-d_2)$$

proof: ① nash-solution is unique
② Nash-satisfies the 4 properties
③ every ϕ satisfying 4 properties $\Rightarrow$ Nash $= \phi$

wlog $(S,d) \to (S-d, 0)$

$$N(S-d,0) = \underset{\bar{s}\in S-d}{\operatorname{argmax}} \bar{s}_1\bar{s}_2$$

if not true, $\exists$ x and y s.t $x_1x_2 = y_1y_2$



$y_2 > x_2$ $y_1 < x_1$

let $\bar{s} = \frac{1}{2}(x+y) \in S-d$ ($\because$ convex)

$$\Rightarrow \bar{s}_1\bar{s}_2 = \left(\frac{1}{2}(x_1+y_1)\right)\left(\frac{1}{2}(x_2+y_2)\right)$$

$$= \frac{1}{4}\left[x_1x_2 + y_1y_2 + x_1y_2 + y_1x_2\right]$$

$x_1x_2 = A+B$ $\quad y_1y_2 = A+C$ $\quad x_1y_2 = A+B+C+D$
$y_1x_2 = A$

$$= \frac{1}{4}\left[(A+B)+\underbrace{(A+B)}_{A+C=A+B}+(A+B+B)+D+A\right]$$

$$= (A+B)+\frac{D}{4} > A+B = x_1x_2 = y_1y_2$$

$\Rightarrow$ this is a contradiction

Symm: (S,d) symmetric $\Rightarrow \phi_1 = \phi_2$ $(y_1^*-d)(y_2^*-d)$ true if (y_1^*, y_2^*) is max $\Rightarrow (y_2^*, y_1^*)$ also max

as unique $\Rightarrow y_1^* = y_2^*$

Eff: as max, otwise will improve

Cpat: $(S,d) \longrightarrow (as+b, ad+b)$

$\uparrow_{y^*}$

$$(y_1^*, y_2^*) = \underset{(x_1,x_2)\in S}{\operatorname{argmax}} [(ax_1+b-ad_1-b)][(ax_2+b-ad_2-b)]$$

$$\Rightarrow (ay_1^*+b, ay_2^*+b)$$

$$= \underset{aS+b}{\operatorname{argmax}}\ (\ )(\ )$$

IIA: the optimising set is getting smaller and optimal point lies in smaller set

now if ϕ satisfy all 4 properties then using PAT more

$d \to (0,0)$

$y^* = N$ as optimal POINT

let $y^* \to (1,1)$

now, $\phi(S,d) = y^*$ then we are done

as $N(S,d) = y^*$

i.e $N(S',0) = (1,1)$

↙ as $\exists$ atleast one $x \in S$ s.t $x > d \Rightarrow y^* > d$

let $L(x_1,x_2) = \left(\frac{x_1-d_1}{y_1^*-d_1}, \frac{x_2-d_2}{y_2^*-d_2}\right)$ $x \in S$

↑ transformation

$L(d_1,d_2) = (0,0)$

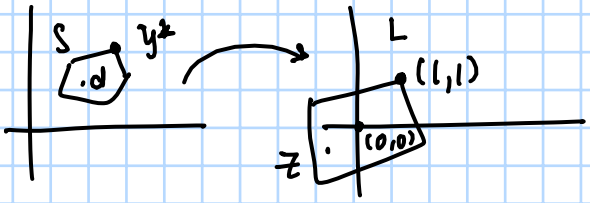
$L(y_1,y_2) = (1,1)$

now, $x_1+x_2 \leq 2$ $\forall$ $x \in \underbrace{aS+b}_{\text{News}} = L$

as if not then $\exists z \in L$ s.t $z_1+z_2 > 2$

$(1,1)$ maximizes nash product

as S is convex, L is also convex



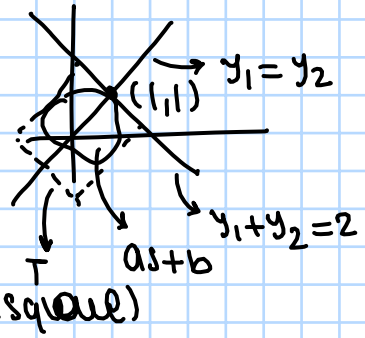
let $(1-\varepsilon)(1,1)+(\varepsilon)(z)=(z_1,z_2)$
some $\varepsilon>0$ close to 0

$$z_1 z_2 = (1-\varepsilon+\varepsilon z_1)(1-\varepsilon+\varepsilon z_2)$$
$$= (1-\varepsilon)^2+(1-\varepsilon)(\varepsilon)(z_1+z_2)+\varepsilon^2 z_1 z_2$$
$$> (1-\varepsilon)(\varepsilon)2+(1-\varepsilon)^2+\varepsilon^2 z_1 z_2$$
$$> 1$$
for $\varepsilon \to 0$

this is a contradiction

so, $x_1+x_2 \leq 2 \ \forall x \in L$

then as S is bounded $\Rightarrow L$ is bounded



if we choose a square symmetric to $y_1=y_2$ and one side along $y_1+y_2=2$ line

then $as+b \subseteq T$ and by symmetry of ϕ and efficiency of ϕ

$$\phi(T,(0,0)) = (1,1)$$

ϕ also satisfies IIA $\Rightarrow \phi(as+b,(0,0))=(1,1)$

$\Rightarrow \phi(S,d) = y^*$ by CPAT

$\Rightarrow \phi(S,d) = N(S,d)$